

Immersing complete digraphs

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Abstract

We consider the problem of immersing the complete digraph on t vertices in a simple digraph. For Eulerian digraphs, we show that such an immersion always exists whenever minimum degree is at least $t(t-1)$, and for $t \leq 4$ minimum degree at least $t-1$ suffices. On the other hand, we show that there exist non-Eulerian digraphs with all vertices of arbitrarily high in- and outdegree which do not contain an immersion of the complete digraph on 3 vertices. As a side result, we obtain a construction of digraphs with large outdegree in which all cycles have odd length, simplifying a former construction of such graphs by Thomassen.

1 Introduction

In this paper all digraphs are finite and may have loops and multiple edges. A digraph D is *simple* if D has no loops and there is at most one edge from x to y for any $x, y \in V(D)$. Note that oppositely oriented edges between two vertices are allowed in simple digraphs, and such a pair of edges is called a *digon*. The *complete*

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digraph of order t , denoted \vec{K}_t , is a digraph with t vertices and a digon between each pair of vertices. The rest of our terminology is fairly standard, as used in [1] or [2].

A directed path of length two from a vertex x to a vertex z in a digraph D , say xyz , can be *split off* by deleting the directed edges xy and yz and adding the directed edge xz (possibly in parallel to existing directed edges). A digraph F is said to be *immersed* in a digraph D if (a digraph isomorphic to) F can be obtained from a subgraph of D by splitting off directed paths of length two (and deleting isolated vertices). Equivalently, F is immersed in D if there exists an injective map $\varphi : V(F) \rightarrow V(D)$ and a collection of edge-disjoint directed paths in D , one from $\varphi(u)$ to $\varphi(v)$ for every edge uv in F . In this setting, the vertices $\{\varphi(v) : v \in V(F)\}$ are called *terminals*. When the collection of directed paths for an immersion is internally disjoint from its set of terminals, the immersion is said to be *strong*; otherwise it may be called *weak*. We omit these descriptors in this paper, as we are only interested in the weaker notion.

Our main interest is in finding conditions on a digraph which imply the existence of certain immersions. Although this appears to be a very natural problem it seems to have received rather little attention. One exception to this is a result of Mader [11], who proved that every digraph D of minimum outdegree t immerses the digraph consisting of two vertices x, y and t copies of the edge from x to y (or, equivalently, D has t edge-disjoint directed paths from u to v for some $u, v \in V(D)$). In contrast to this, no outdegree assumption implies the existence of \vec{K}_2^2 – the digraph consisting of two vertices x, y and two copies of each non-loop edge xy and yx ; see Figure 1. This was first proved by Mader [10], who utilized a family of digraphs constructed earlier by Thomassen [14] which have high outdegree but no cycle of even length. We give a new proof of this result (highlighted below) which is based on a simpler construction. In fact, our graphs also give an easy example of high outdegree digraphs with no even cycle, thus simplifying the former construction by Thomassen. Since this latter property is not our focus, this argument is relegated to an appendix.

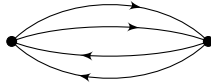


Figure 1: The digraph \vec{K}_2^2

Theorem 1 *For every positive integer k there exists a simple digraph D with minimum in- and outdegree at least k so that D does not immerse \vec{K}_2^2 .*

Mader also considered the problem of finding subdivisions of transitive tournaments. He proved in [12] that every digraph of minimum outdegree 3 contains a subdivision of the transitive tournament on 4 vertices, and he conjectured in [10]

that there exists a function f so that every digraph of minimum outdegree $f(t)$ contains a subdivision of the transitive tournament on t vertices. Weakening the conclusion of this conjecture to allow for an immersion of the transitive tournament on t vertices yields the following interesting open problem.

Conjecture 2 *There exists a function f so that every simple digraph of minimum outdegree $f(t)$ contains an immersion of the transitive tournament on t vertices.*

In light of Theorem 1 we cannot hope to find a subdivision or even an immersion of a complete digraph \vec{K}_t for $t \geq 3$ without some assumption beyond a minimum degree condition. One positive result of this type is a theorem of Kühn, Osthus and Young [7] which shows that every suitably dense digraph contains a subdivision of a large complete digraph. Our main results show that a minimum degree condition together with the added assumption of Eulerian implies the existence of a \vec{K}_t immersion. Given an Eulerian digraph, we will use the term *degree* to refer to both the outdegree and the indegree of a vertex, and similarly for the terms *minimum degree* and *maximum degree*.

Theorem 3 *Every simple Eulerian digraph with minimum degree at least $t(t-1)$ contains an immersion of \vec{K}_t .*

The quadratic bound of Theorem 3 can be strengthened for small values of t as follows.

Theorem 4 *For $t \leq 4$, every simple Eulerian digraph of minimum degree at least $t-1$ contains an immersion of \vec{K}_t .*

In light of Theorem 3, we may define a function f by the rule that $f(t)$ is the smallest integer so that every simple Eulerian digraph of minimum degree at least $f(t)$ contains an immersion of \vec{K}_t . Our results (together with the trivial lower bound) show that $t-1 \leq f(t) \leq t(t-1)$ and $f(t) = t-1$ for $t \leq 4$.

For ordinary (undirected) graphs, there is an analogous function g defined by the rule that $g(t)$ is the smallest integer so that every graph of minimum degree at least $g(t)$ contains an immersion of K_t (note that we have dropped the assumption of Eulerian here). Our understanding of this function is considerably better. In particular, a recent theorem established by the current authors together with Fox and Dvořák [4] shows that $g(t) \leq 200t$ so $g(t) = \Theta(t)$. Further, a theorem of Lescure and Meyniel [8] (see also DeVos et al. [3]) shows that $g(t) = t-1$ for $t \leq 7$, while an example due to Seymour shows that $g(t) \geq t$ for $t \geq 10$ (see [3] or [8]).

Although high in and outdegree at every vertex does not imply the existence of an immersion of \vec{K}_3 in a general digraph, it is possible to force the existence of such an immersion under a connectivity assumption. We say that a digraph D is

strongly k -edge-connected if $D - S$ is strongly connected for every $S \subseteq E(D)$ with $|S| < k$. For a digraph D and a vertex $v \in V(D)$ we define an *arborescence with root v* to be a subdigraph T of D which is a spanning tree in the underlying graph of D and has the property that all edges are directed “away” from v . Note that v is the unique vertex of T with indegree 0, and that all other vertices of T have indegree 1. Edmonds’ Disjoint Arborescence Theorem ([5], [6]; see [13] for a proof) gives a necessary and sufficient condition for the existence of a collection of edge-disjoint arborescences with prescribed root vertices in a general digraph.

For a digraph D and a set $X \subseteq V(D)$ we let $d^+(X)$ denote the number of edges with initial point in X and terminal point in $V \setminus X$ and we set $d^-(X) = d^+(V \setminus X)$.

Theorem 5 (Edmonds’ Disjoint Arborescence Theorem [5], [6]) *Let D be a digraph, and let $v_1, \dots, v_\ell \in V(D)$ (not necessarily distinct). Then there exist edge-disjoint arborescences T_1, \dots, T_ℓ so that T_i has root v_i if and only if every $X \subset V(D)$ satisfies the following condition:*

$$d^+(X) \geq |\{i : v_i \in X, 1 \leq i \leq \ell\}|.$$

This result has the following easy corollary.

Corollary 6 *If D is a strongly $t(t-1)$ -edge-connected digraph with $|V(D)| \geq t$ then D contains an immersion of \vec{K}_t .*

Proof. Choose distinct vertices v_1, \dots, v_t and apply Edmonds’ Disjoint Arborescence Theorem to choose $t(t-1)$ edge-disjoint arborescences with exactly $t-1$ of them having root v_i for $1 \leq i \leq t$. For each $1 \leq i \leq t$ use the $t-1$ arborescences rooted at v_i to find a set of edge-disjoint paths from v_i to each other vertex v_j . All together, these paths give an immersion of \vec{K}_t . \square

The following result shows that the connectivity condition of Corollary 6 is best possible up to a factor of 2.

Theorem 7 *For every positive integer $t \geq 3$ there exists a simple digraph which is strongly $\frac{1}{2}t(t-3)$ -edge-connected and does not immerse \vec{K}_t .*

Continuing in the vein of our earlier analysis, we define a function h by the rule that $h(t)$ is the smallest integer so that every simple digraph of strong edge-connectivity at least $h(t)$ immerses \vec{K}_t . Corollary 6 together with Theorem 7 then shows that $\frac{1}{2}t(t-3) + 1 \leq h(t) \leq t(t-1)$.

Note that the upper and lower bounds on $h(t)$ differ only by a factor of approximately 2, while our bounds on $f(t)$ differ by a factor of t when $t \geq 5$. It would be of significant interest to obtain better bounds on this function f . In Section 4

of this paper we establish the exact values of $f(t)$ when $t \leq 4$ (Theorem 4), and offer an example indicating some of the difficulty involved in extending our methods to determine $f(5)$. The proof of the upper bound on f , Theorem 3, is the subject of Section 3. This proof uses Corollary 6 as well as a structural result showing that every simple Eulerian digraph of minimum degree $t(t-1)$ immerses a strongly $t(t-1)$ -edge-connected digraph on at least t vertices. In the following section we construct examples to prove the lower bound on $h(t)$ (Theorem 7), as well as the fact that arbitrarily high in- and outdegrees do not necessarily imply even a \vec{K}_3 immersion (Theorem 1). As we will see, the former examples are actually extensions of the latter.

2 Examples

We construct, for every positive integer k , a simple digraph D_k as follows. Take a rooted (undirected) tree of height k where the root vertex has degree k , and every vertex at distance $j \geq 1$ from the root has degree $k-j+1$. Orient the edges of this tree so that it is an arborescence from the root (so all edges are directed “away” from the root). See Figure 2 for an example with $k = 4$. Additionally, for every vertex x (at level $j \geq 1$), add an edge from x to every vertex on the path from the root to x (a total of $j-1$ edges).

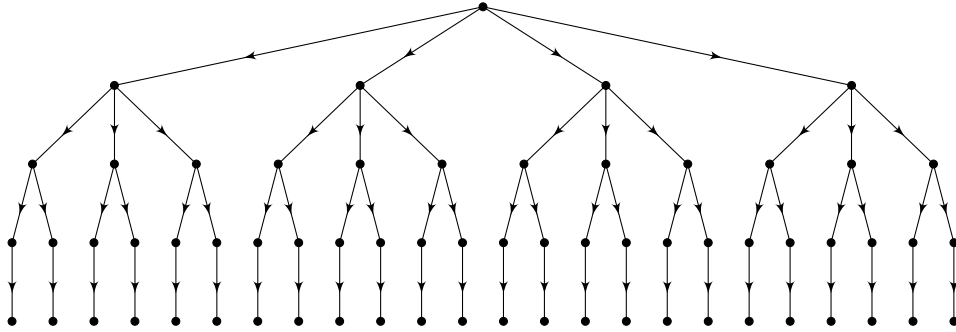


Figure 2: Oriented rooted tree used to construct D_4

For two distinct vertices x, y in a digraph D we define $\lambda(x, y)$ to be the minimum of $d^+(X)$ over all subsets $X \subseteq V(D)$ with $x \in X$ and $y \notin X$. Menger’s Theorem implies that $\lambda(x, y)$ is also equal to the maximum size of a collection of edge-disjoint directed paths from x to y .

Observation 8 *For every $k \geq 1$ the digraph D_k has the following properties.*

1. *Every vertex in D_k has outdegree k .*

2. For every $u, v \in V(D_k)$ with $u \neq v$ we have $\min\{\lambda(u, v), \lambda(v, u)\} \leq 1$.
3. For every $v \in V(D_k)$ there exist k edge-disjoint directed paths starting at v with exactly one ending at each vertex at level 1 (i.e. the outneighbours of the root).

Proof. Property 1 follows immediately from the definition. For property 2, note that there is an edge-cut separating u and v which intersects the rooted tree in a single edge e . Since every edge other than e has the opposite orientation to e in this cut, we must have $\min\{\lambda(u, v), \lambda(v, u)\} \leq 1$. For property 3, note that every vertex other than the root has an edge to the root. So, for a non-root vertex v , there are exactly k directed paths of length ≤ 2 from v to the root and these are all edge-disjoint. Extending each of these along a different edge away from the root gives a collection of k edge-disjoint walks, which implies the existence of the desired paths. \square

Proof of Theorem 1. Let D'_k be a copy of D_k with all edges reversed, and construct the digraph D from the disjoint union of D_k and D'_k by adding all possible edges from D'_k to D_k . It follows from the first part of the observation that all vertices of D have indegree and outdegree at least k . If D contains a \vec{K}_2^2 -immersion, then such an immersion must also exist in D_k , but this would contradict the second part of the observation. \square

Proof of Theorem 7. Since the Theorem is trivially true for $t = 3$, we may assume $t \geq 4$. We begin with the digraph D constructed in the proof of the previous theorem for the parameter $k = \frac{1}{2}t(t - 3)$. Let X denote the outneighbours of the root vertex in D_k and let X' denote the inneighbours of the root vertex in D'_k . Form the digraph F from D by adding a perfect matching M between X and X' so that all edges are oriented from X to X' . The proof of the theorem will follow from two separate claims.

Claim F is strongly k -edge-connected.

To check this, we let $u, v \in V(F)$, and we shall show that there exist k edge-disjoint walks from u to v . If $u, v \in D_k$ then we may use part 3 of the observation to choose k edge-disjoint paths in D_k starting at u and ending at X . Now, extend each of these along an edge of M to obtain edge-disjoint walks ending at X' and then extend each of these walks with endpoint x' along the edge $x'v$ to obtain k edge-disjoint walks from u to v . A similar argument works for $u, v \in D'_k$. If $u \in D_k$ and $v \in D'_k$ then (again using part 3 of the observation) we may choose k edge-disjoint paths from u to X in D_k and k edge-disjoint paths from X' to v in D'_k . By adding the edges in our matching M to these, we obtain k edge-disjoint paths from u to v as desired. Finally, if $u \in D'_k$ and $v \in D_k$ then there exist k edge-disjoint single edge paths from u to X , these may be extended using the edges of M to k edge-disjoint

paths from u to X' and now since every vertex in X' has an edge to v , they can be extended to give the desired paths.

Claim F does not contain an immersion of \vec{K}_t .

Suppose (for a contradiction) that F contains an immersion of \vec{K}_t . Since each edge of M can only help in forming a single edge in this immersed digraph, it follows that $F - M$ must immerse a digraph K' obtained from \vec{K}_t by deleting k edges. Since K' still has at least $\binom{t}{2} - k = \frac{1}{2}t(t-1) - \frac{1}{2}t(t-3) = t$ digons, it follows that K' must contain a subdigraph which is isomorphic to the digraph obtained from a cycle (of length ≥ 3) by replacing each edge by a digon. If u, v are vertices in this subdigraph, then we must have $\lambda_{F-M}(u, v) \geq 2$ and $\lambda_{F-M}(v, u) \geq 2$. However, this contradicts part 2 of the observation. \square

3 A quadratic bound

The goal of this section is to prove Theorem 3. Our proof will use the corollary of Edmonds' Disjoint Arborescence Theorem stated in the introduction, and also the following classical result of Mader on splitting.

Theorem 9 (Mader's Directed Splitting Theorem [9]) *Let D be a digraph with a distinguished vertex v . Suppose that $\deg^+(v) = \deg^-(v)$ and the following is satisfied:*

(\star) *Every nonempty $X \subset (V(D) \setminus \{v\})$ satisfies $d^+(X) \geq k$ and $d^-(X) \geq k$.*

Then for every edge uv there exists an edge vw so that the new digraph formed by splitting off the path uvw still satisfies property (\star).

With this, we are ready to prove our main structural result from this section. Extending our earlier notation, for a digraph D and a set $X \subseteq V(D)$ we let $d(X) = d^+(X) + d^-(X)$. Note that every Eulerian digraph satisfies $d^+(X) = d^-(X)$, so, in particular, $d(X)$ is always even.

Theorem 10 *Every simple Eulerian digraph with minimum degree r immerses a strongly r -edge-connected Eulerian digraph F with $|V(F)| > r$ and the property that $F - S$ is simple for a set $S \subseteq E(F)$ with $|S| < r$*

Proof. Let D be a simple digraph with minimum degree r . Set $X_{-1} = V(D)$ and choose a sequence of subsets X_0, \dots, X_{r-1} according to the rule that, for $0 \leq k \leq r-1$:

X_k is a minimal subset of X_{k-1} with the property that $d(X_k) \leq 2k$.

Note that $d(X_{-1}) = 0$ and more generally $d(X_{k-1}) \leq 2k$ holds for all $k \geq 0$ by induction, so the above choice is always possible. Note further that every X_k with $k \geq 0$ induces a connected subdigraph.

Claim *If $x, y \in X_k$ and $x \neq y$ then $\lambda_D(x, y) \geq k + 1$.*

To see this, suppose (for a contradiction) that it fails, and choose a set $Z \subseteq V(D)$ so that $|Z \cap \{x, y\}| = 1$ and $d^+(Z) \leq k$ (note this last condition is equivalent to $d(Z) \leq 2k$ since D is Eulerian) and subject to this the smallest index j so that $Z \not\subseteq X_j$ is as large as possible. We cannot have $Z \subseteq X_k$ as otherwise Z would already contradict the choice of X_k , and thus j exists. Now, setting $\bar{Z} = V(D) \setminus Z$ and using the submodularity of the function d gives us

$$d(X_j \setminus Z) + d(Z \setminus X_j) = d(X_j \cap \bar{Z}) + d(X_j \cup \bar{Z}) \leq d(X_j) + d(\bar{Z}) \leq 2j + 2k.$$

The set $Z \setminus X_j$ is a proper nonempty subset of X_{j-1} so by assumption we have $d(Z \setminus X_j) \geq 2j$. But then the above inequality implies that the set $Z' = X_j \setminus Z$ satisfies $d(Z') \leq 2k$ and then Z' contradicts the choice of Z , thus completing the proof of the claim.

If $X_{r-1} = X_0$ then D has a strongly r -edge-connected component, and we are already finished. Otherwise, choose the smallest integer j so that $X_{r-1} = X_j$ and note that $j \geq 1$ and $d(X_j) = 2j$ (were $d(X_j) < 2j$ we would have $X_j = X_{j-1}$). Now, choose a vertex $x \in X_j$ and a vertex $y \in X_{j-1} \setminus X_j$. It follows from the claim that $\lambda_D(x, y) \geq j$ so we may choose a collection P_1, \dots, P_j of edge-disjoint directed paths from x to y . It follows from the assumption that D is Eulerian that the digraph obtained from D by removing the edges of these paths has a collection Q_1, \dots, Q_j of edge-disjoint directed paths from y to x . Together, the existence of P_1, \dots, P_j and Q_1, \dots, Q_j implies that D immerses the digraph D' obtained from D by identifying $V \setminus X_j$ to a single new vertex w and deleting any resulting loops. Since this identification can only increase the function λ and since $X_j = X_{r-1}$ we then have $\lambda_{D'}(u, v) \geq \lambda_D(u, v) \geq r$ for every $u, v \in X_{r-1}$. Now, we repeatedly apply Mader's Theorem to do splits at the vertex w of D' until w becomes an isolated vertex, and we let F be the component of the resulting digraph with vertex set X_{r-1} . We now have $\lambda_F(u, v) \geq r$ for every $u, v \in X_{r-1}$ with $u \neq v$ so F is strongly r -edge-connected. Furthermore, setting S to be the set of edges in F formed by doing splits at w , we find that $|S| < r$ and $F - S$ is simple. To complete the proof, set $n = |V(F)| \geq 2$ and note that $F - S$ has at most $n(n-1)$ edges which gives us $n(n-1) + r > |E(F)| = \sum_{v \in V(F)} \deg_F^+(v) \geq rn$. This implies that $n > r$ as desired. \square

Proof of Theorem 3. We may assume that $t \geq 2$ as otherwise the result is trivial. By Theorem 10 we may choose a digraph D' immersed in D which is strongly $t(t-1)$ -edge-connected and has at least $t(t-1) \geq t$ vertices. It now follows from Corollary 6 that D' contains an immersion of \vec{K}_t which completes the proof. \square

4 Immersing small complete digraphs

The following lemma shows that if we want to immerse \vec{K}_t in an Eulerian digraph D , then we may as well assume that D is regular.

Lemma 11 *Let t be a positive integer, and let D be a simple Eulerian digraph with minimum degree at least t . Then D contains an immersion of a simple t -regular Eulerian digraph.*

Proof. If $\Delta(D) = t$, then D itself is t -regular and we are done. So we may assume $\Delta = \Delta(D) > t$. Let $v \in V(D)$ be a vertex with $\deg(v) = \Delta$. Since an inneighbour u of v has degree at most Δ , there are at most $\Delta - 1$ edges from u to outneighbours of v . Hence, either v has an inneighbour u and an outneighbour $w \neq u$ with no edge from u to w , or all inneighbours of v are also outneighbours of v and all possible edges between the neighbours of v are present. In the first case we split off uvw and use induction on a component of the resulting graph. In the second case we are done because D contains \vec{K}_{t+1} as a subgraph. \square

We are now ready to show that both \vec{K}_3 and \vec{K}_4 immerse as we would like.

Proof of Theorem 4. For $t \leq 2$ the result is trivial, since an Eulerian digraph is strongly connected, and we only look for a cycle in D . Thus we only have to deal with $t = 3$ and $t = 4$. By Lemma 11 it suffices to prove that any $(t - 1)$ -regular simple Eulerian digraph contains an immersion of \vec{K}_t . In fact, we will prove the following slightly stronger statement. Note that here, by *parallel class of edges*, we mean a set of edges between two vertices that are oriented in the same direction.

Let $t \in \{3, 4\}$, let D be an Eulerian digraph, and let $v_0 \in V(D)$. Assume that D satisfies the following conditions:

- $|V(D)| \geq 2$.
- $\deg(v_0) \leq t - 1$ and $\deg(v) = t - 1$ for all $v \in V(D) \setminus \{v_0\}$.
- *There is at most one parallel class of edges, and it is incident with v_0 .*

Then D contains an immersion of \vec{K}_t .

Assume (for a contradiction) that D is a counterexample with $|V(D)| + |E(D)|$ minimum. We call v_0 the *exceptional vertex*. We will prove four properties of D before deducing a contradiction.

Claim 1 *If v_0 has an inneighbour u and an outneighbour w different from u , then there is an edge from u to w .*

If Claim 1 is false, then let D' be the graph obtained from D by splitting off uvw . Clearly the component of D' containing u and w is a smaller counterexample,

where the exceptional vertex is either v_0 (if v_0 is in that component) or else any vertex.

Claim 2 *There is no parallel class.*

If there is a parallel class, it has to be incident with v_0 . By symmetry, we may assume that v_0 has two (or three) parallel in-edges from a vertex u . Since $\deg(u) \leq t - 1 \leq 3$, it follows from Claim 1 that v_0 has at most one outneighbour different from u . Given that there is only one parallel class, this means that $\deg(v_0) = 2$ and hence $N^+(v_0) = \{u, w\}$ for some vertex $w \neq u$. Note that, by Claim 1, there is an edge uw , thus $\deg(u) = 3$ and we are in the case $t = 4$. Now let D' be obtained from D by splitting off uv_0w and deleting v_0 afterwards. Clearly, any immersion in D' extends to an immersion in D . In D' all vertices except u have degree three, and there is one parallel class between u and w . Thus D' (with new exceptional vertex u) is a smaller counterexample.

Claim 3 $\deg(v_0) \geq 2$.

Since $|V(D)| \geq 2$ and D is Eulerian we know that $\deg(v_0) > 0$. If $\deg(v_0) = 1$ with inneighbour u and outneighbour w , then either $u = w$ (in which case $D - \{v_0\}$ with new exceptional vertex u is a smaller counterexample), or $u \neq w$ (in which case splitting v_0 results in a smaller counterexample with new exceptional vertex u or w). Note that in the former case, we maintain the required property of having at least two vertices since $u = w$ has degree $t - 1 \geq 2$ in D .

Claim 4 $\deg(v_0) < t - 1$.

If $\deg(v_0) = t - 1$ then, by Claim 2, v_0 has $t - 1$ distinct outneighbours. The inneighbours of v_0 have to be identical to the outneighbours, otherwise there would be a vertex of degree at least t by Claim 1. Hence, again by Claim 1, v_0 and its $t - 1$ neighbours form a \vec{K}_t , a contradiction.

By Claims 3 and 4 imply that $t = 4$ and $\deg(v_0) = 2$. By Claim 2, v_0 has two distinct inneighbours, say v_1 and v_2 . We distinguish three cases, based on where the two outneighbours of v_0 are.

Case 1 $N^+(v_0) = \{v_1, v_2\}$.

By Claim 1, v_1 and v_2 are bidirectionally joined. For $i = 1, 2$, let u_i be the remaining inneighbour of v_i , and let w_i be the remaining outneighbour of v_i . Now let D' be the digraph obtained from D by contracting the vertices v_0, v_1, v_2 to a single vertex v'_0 . It is easy to see that any immersion in D' extends to an immersion in D . In D' all vertices except v'_0 have degree three. If $u_1 \neq u_2$ or $w_1 \neq w_2$ then D' contains at most one parallel class, thus D' (with exceptional vertex v'_0) is a smaller counterexample. Hence we may assume that $u_1 = u_2$ and $w_1 = w_2$. If $u_1 = w_1$ then $D' - \{v'_0\}$ is a smaller counterexample (with new exceptional vertex $u_1 = w_1$).

Otherwise let D'' be the digraph obtained from D' by splitting off all edges incident to v'_0 . Note that D'' is immersed in D , so a \vec{K}_4 -immersion in D'' does not exist. Since D'' is 3-regular and contains exactly one parallel class from u_1 to w_1 , it is a smaller counterexample.

Case 2 $|N^+(v_0) \cap \{v_1, v_2\}| = 1$, say $v_2 \in N^+(v_0)$.

Let v_3 be the remaining outneighbour of v_0 . By Claim 1, we have edges from v_1 to v_2 and to v_3 , and from v_2 to v_3 . Hence, v_2 has one additional inneighbour $u \notin \{v_0, v_1\}$ and one additional outneighbour $w \notin \{v_0, v_3\}$. First consider the case that $u = v_3$ and $w = v_1$. Then the coboundary of $V_0 = \{v_0, v_1, v_2, v_3\}$ consists of exactly two in-edges at v_1 and two out-edges at v_3 . Now let D' be the graph obtained from D by contracting V_0 to a single vertex v'_0 . Since $D[V_0]$ contains two edge-disjoint paths from v_1 to v_3 , any immersion in D' extends to an immersion in D . All vertices except v'_0 have degree three in D' , and there is no parallel class in D' . Thus D' is a smaller counterexample. Hence we have $u \neq v_3$ or $w \neq v_1$. By symmetry, we may assume $u \neq v_3$. Now let D' be the graph obtained from D by splitting off uv_2v_3 , v_0v_2w , and $v_1v_2v_0$, and removing v_2 . Since $u \notin \{v_0, v_1, v_3\}$ and $w \neq v_3$, this split creates no loops and gives only one parallel class between v_1 and v_0 . Since all degrees (except for v_2) stay the same, D' (with exceptional vertex v_0) is a smaller counterexample.

Case 3 $N^+(v_0) \cap N^-(v_0) = \emptyset$.

Let $N^+(v_0) = \{v_3, v_4\}$. By Claim 1, we have all edges from v_1 and v_2 to v_3 and v_4 . Let u_1, u_2, u_3 be the three inneighbours of v_1 . We may assume that one of them, say u_3 , is not v_4 , and one of the other two, say u_2 is not v_3 . Also note that $v_2 \notin \{u_1, u_2, u_3\}$, because we know all three outneighbours of v_2 , and none of them is v_1 . Now let D' be the graph obtained from D by splitting off $u_1v_1v_0$, $u_2v_1v_3$, and $u_3v_1v_4$. Since $u_1 \notin \{v_0, v_2\}$, $u_2 \notin \{v_0, v_2, v_3\}$ and $u_3 \notin \{v_0, v_2, v_4\}$, no loops or parallel edges are created. Hence, D' is a smaller counterexample. \square

One would hope that proof of Theorem 4 could be generalized to larger \vec{K}_t , in particular to \vec{K}_5 . Consider the following observation about this immersion problem: If every simple Eulerian digraph D with minimum degree d immerses \vec{K}_t , then the same conclusion must still hold under the weaker assumption that D has all but one vertex of degree at least d . To see this, note that were D to be a counterexample to this stronger property with the single vertex v of degree less than d , then taking d disjoint copies of D and identifying all copies of the vertex v yields a digraph of minimum degree d which still does not immerse \vec{K}_t . In light of this, it is natural to permit one exceptional vertex of low degree. However, the digraph below indicates some of the difficulty in using the approach of Theorem 4 for \vec{K}_5 .

In the digraph of Figure 3, all vertices have degree at least 4 except for the vertex v_0 , but however we choose a vertex and split off all of its edges, we move to

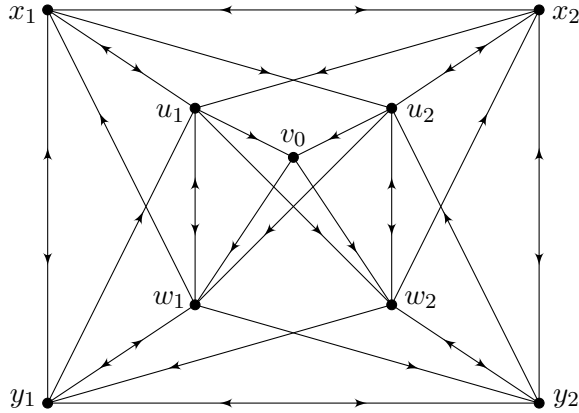


Figure 3: A difficult configuration for trying to immerse \vec{K}_5

a new digraph which has at least two nontrivial disjoint parallel classes. Hence the structured assumption used to prove Theorem 4 does not extend easily for immersing larger complete digraphs.

Appendix

Here we give a simplified proof of the following result:

Theorem 12 (Thomassen [14]) *For every k , there exists a digraph with minimum outdegree at least k in which each cycle is of odd length.*

Our graphs used to prove Theorem 12 are quite closely related to those constructed in Section 2, and the following lemma contains the key property we require.

Lemma 13 *Let D be a digraph, let F be an arborescence of D and assume that for every edge $xy \in E(D) \setminus E(F)$ there exists a directed path in F from y to x . Then every directed cycle in D contains exactly one edge in $E(D) \setminus E(F)$.*

Proof. Let C be a directed cycle in D and choose an edge $xy \in E(C) \setminus E(F)$. Set P to be the directed path from y to x in F . Now, for every edge e in P , the fundamental cut of F with respect to this edge separates x and y and has the property that all edges other than e in this cut are in the opposite direction to e . It follows from this that e must appear in the cycle C . But then C must consist of precisely the edges in P together with xy . \square

Based on this simple lemma, we can easily construct digraphs of large outdegree

without directed cycles of even length. Let F be an arborescence with the property that all leaf vertices are distance $2k$ from the root and all non-leaf vertices have outdegree exactly k . Now, we add edges to F to form a new digraph D by the following rule. For every directed path in F of even length from a vertex y to a leaf vertex x , we add the edge xy to D . It is immediate that every vertex in D has outdegree k , and it follows from the above lemma that it has no directed cycle of even length.

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